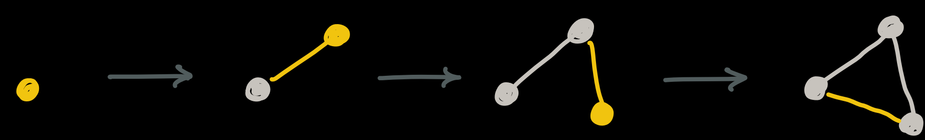
07/11/22

5. Polytopal Complexes & Shellability

- Our immediate goal: proving the Euler-Poincaré identity

$$-f_{-1}+f_0-f_1+f_2-\cdots+(-1)^d f_d = 0$$

- Recall proof of 3D-case: $V-E+F=2$
  by induction:
  - build planar graph vertex-by-vertex / edge-by-edge
  - check identity for single-vertex graph
  - check that each step preserves identity



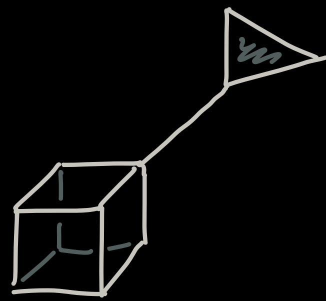
- Can we as well build higher-dimensional polytopes piece-by-piece? Maybe facet-by-facet?
  - What are the objects we encounter on the way?
  - not quite polytopes, since not "closed up" yet

  → polytopal complexes

5.1 Polytopal complexes

(1)

(2)



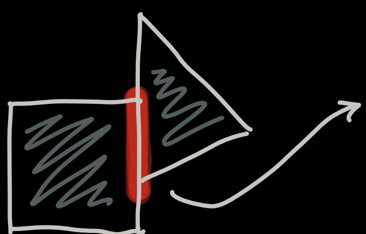
"polytopes glued together along faces"

Def: A **(polytopal) complex** $\mathcal{C}$ is a family of polytopes $P_1, \ldots, P_m \subset \mathbb{R}^d$ so that

not necessarily finit, but finite is sufficient for our purpose

(i) if $P \in \mathcal{C}$ and $f \in \mathcal{F}(P) \longrightarrow f \in \mathcal{C}$

(ii) if $P, Q \in \mathcal{C} \longrightarrow P \cap Q$ is a face of both P and Q.



not a common face
not a complex

One uses terminology close to polytopes

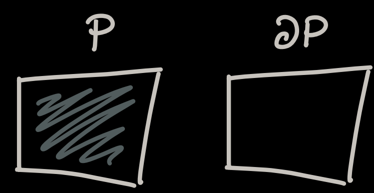
- elements of $\mathcal{C}$ are called **faces**
- highest-dimensional faces are called **facets**
- the **dimension** of $\mathcal{C}$ is the dimension of a facet

Def: $\mathcal{C}$ is **pure** if every face lies in a facet

See examplex above: (1) is pure, (2) is not

- each polytope P can be considered as a complex $\mathcal{C} := \mathcal{F}(P)$
- the **boundary complex** of $P \subset \mathbb{R}^d$ is

$$\partial P := \mathcal{F}(P) \setminus \{P\}.$$



This is a pure $(d-1)$-complex

<u>Idea for proving Euler-Poincaré</u>:

- complexes have f-vectors
- for a complex $\mathcal{C}$ one might ask for the value of

$$\chi(\mathcal{C}) := -f_{-1} + f_0 - f_1 + \cdots + (-1)^d f_d .$$

"reduced Euler characteristic"

- let's build a polytope facet-by-facet by enumerating the facets $F_1, \ldots, F_m$
- determine $\chi(F_1)$
- check that adding a facet keeps identity valid

BUT: it turns out there are right and wrong ways to enumerate the facets!

$\longrightarrow$ order matters

5.2. Shellings

$\longrightarrow$ the right way to order polytope facets

The following definition is recursive

Def: a **shelling** is an enumeration $F_1, \dots, F_m \in \mathcal{F}_{d-1}(P)$ of the facets of ∂P (works with any pure complex) so that either

(i) the F_i are points (i.e. P is a line segment $\to$ order does not matter)

or

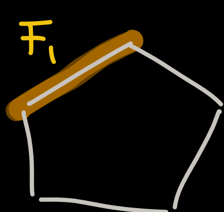
(ii) for all $i \in \{2, \dots, m\}$: $F_i \cap (F_1 \cup \dots \cup F_{i-1})$ is non-empty and an initial segment of a shelling of ∂F_i

NOTE: – ∂F_i is shellable

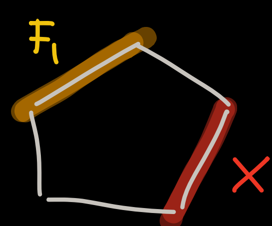
– $F_i \cap (F_1 \cup \dots \cup F_{i-1})$ is pure $(d-2)$-dimensional

Examples:

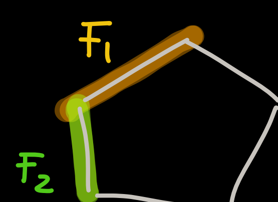
2D:



the first facet can be anywhere

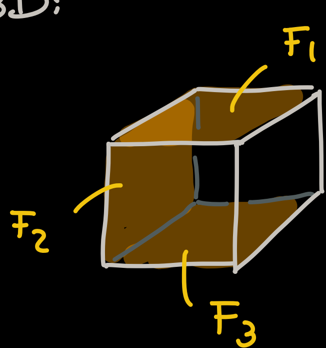


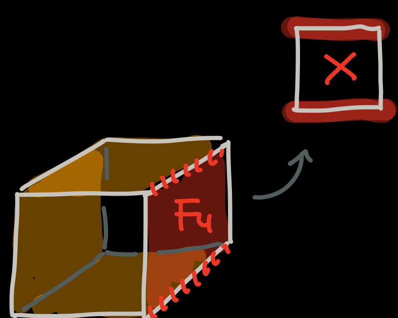
$F_1 \cap F_2$ must be non-empty



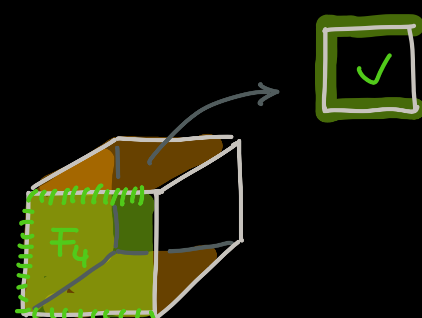
$F_1 \cup \dots \cup F_i$ must be connected at all times

3D:





do not create holes



Q: Do polytopes always have shellings?

$\longrightarrow$ Yes, but let us first see that shellings are indeed the right definition for us.

5.3. Proving the Euler-Poincaré identity

Thm: $P \subset \mathbb{R}^d$, then

- $\chi(P) = -f_{-1} + f_0 - f_1 + \cdots + (-1)^d f_d = 0$

  or equivalently

- $\chi(\partial P) = -(-1)^d$

Proof:

- assume $F_1, \ldots, F_m \in \mathcal{F}_{d-1}(P)$ is a shelling
- we actually show the following:

  (*) $\chi(F_1 \cup \cdots \cup F_i) = 0$ as long as $i < m$,

  and $\chi(F_1 \cup \cdots \cup F_m) = -(-1)^d$ only when we put in the last facet
- we proceed by induction on d

  Ex: verify induction base $d \in \{1, 2\}$
- We need the following

  Claim: $\chi(\mathcal{C} \cup \mathcal{D}) = \chi(\mathcal{C}) + \chi(\mathcal{D}) - \chi(\mathcal{C} \cap \mathcal{D})$

  - since χ is linear in the f-vector, this follows from

    $f(\mathcal{C} \cup \mathcal{D}) = f(\mathcal{C}) + f(\mathcal{D}) - \chi(\mathcal{C} \cap \mathcal{D})$

- taking the union $\mathcal{C} \cup \mathcal{D}$ adds up the face numbers, except where they are "glued together" (in $\mathcal{C} \cap \mathcal{D}$), there we overcount and need to subtract again.

- let's now show (*) by induction on i: ← in $\chi(F_1 \cup \dots \cup F_i)$
  - IB: $i=1 \longrightarrow \chi(F_1) = 0$ by IH$(d-1)$

    Note: we have two intertwined inductions, one on d, one on i.

  - $i \in \{2, \dots, m\}$ then

    Here we need the specific definition of shelling

$$\chi(F_1 \cup \dots \cup F_{i-1} \cup F_i)$$
$$= \underbrace{\chi(F_1 \cup \dots \cup F_{i-1})}_{=0 \text{ by IH}(i-1)} + \underbrace{\chi(F_i)}_{=0 \text{ by IH}(d-1)} - \underbrace{\chi((F_1 \cup \dots \cup F_{i-1}) \cap F_i)}_{\text{initial segment in a shelling of } \partial F_i}$$
$$= -\chi(\text{initial segment in a shelling of } \partial F_i)$$

  - There are two cases:

    $i < m$: the initial segment is proper (not all of ∂F_i)

    $\longrightarrow -\chi(\dots) = 0$ by IH$(d-1)$

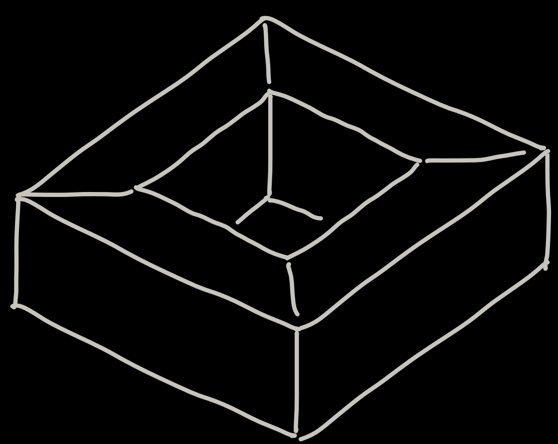
    (one should show this but can be easily seen from the shelling we construct later)

    $i = m$: $F_i \cap (F_1 \cup \dots \cup F_{i-1}) = \partial F_i$

    $\longrightarrow -\chi(\dots) = -\chi(\partial F_i) = -(-(-1)^{d-1}) = -(-1)^d$ by IH$(d-1)$ $\square$

5.4. Existence of Shellings

Not every complex is shellable



Ex: convince yourself that this torus is not shellable

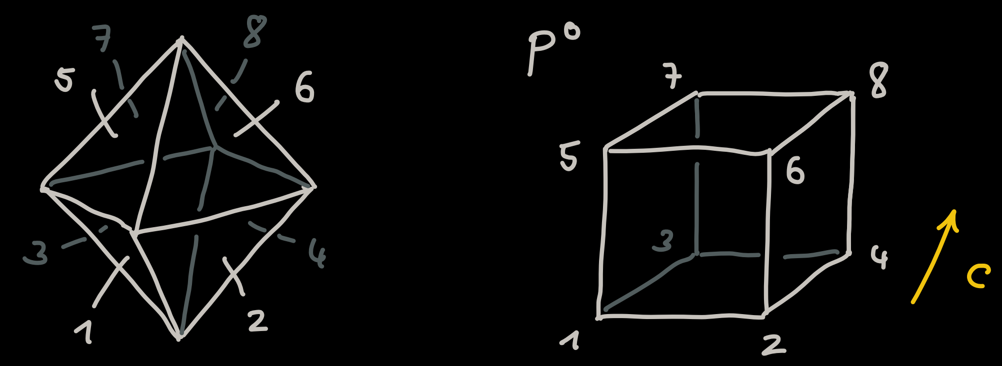
(the problems are the holes)

- one should expect that only "spherical complexes" are shellable (since we proved $\chi = 0$)
- in fact: ∂P is always shellable!
- However, one can get "stuck" while shelling, so doing it naively does not work.

Def: The linear shelling is defined as follows:

- start from $P \subset \mathbb{R}^d$
- look at its polar dual P°
- facets of P correspond to vertices of P°
- choose a generic direction $c \in \mathbb{R}^d$
- order vertices of P° according to $\langle \cdot, c \rangle$

$\longrightarrow$ on facets this is the linear shelling

P: 

In the following assume that $F_1, \dots, F_m$ is the linear shelling, and $v_i \in \mathcal{F}_0(P^\circ)$ corresponds to F_i.

Thm:

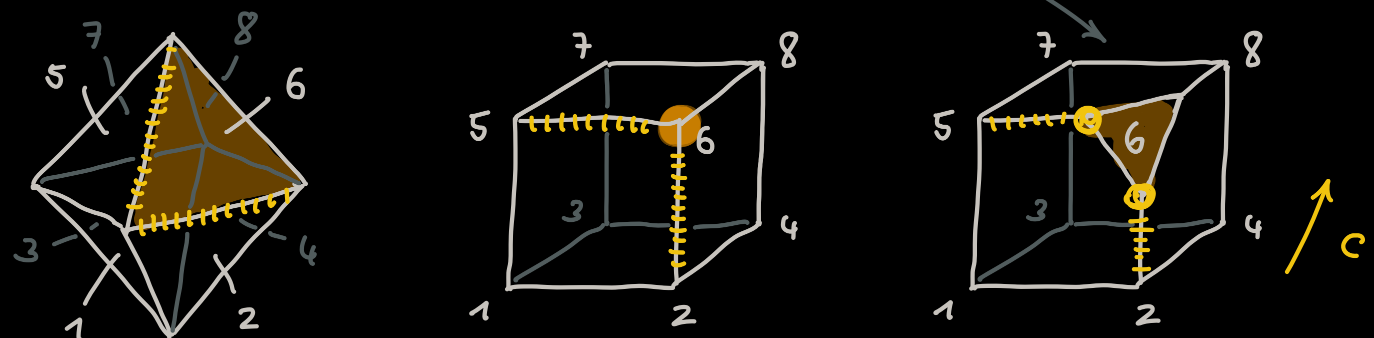
(i) the linear shelling is a shelling of ∂P

(ii) the F_i with $\langle v_i, c\rangle < \kappa$ (for some κ) form an initial segment of a shelling of ∂P

(follows immediately from (i) but important for the inductive proof)

Proof:

- induction on the dimension d of P
- fix a facet F_i, $i \geq 2$ of P
- F_i corresponds to a vertex $v_i \in \mathcal{F}_0(P^\circ)$
- consider the vertex figure P° / v_i



- recall: P°/v_i is dual to F_i
- vertices of P°/v_i correspond to edges of P° incident to v_i, thus to facets of P incident to F_i

- define hyperplane $H: \langle \cdot, c \rangle = \langle v_i, c \rangle$
- vertices of P°/v_i "below" H correspond to both
  1) vertices of P° adjacent to v_i that come before v_i
     $\longrightarrow$ i.e. to the $F_1, \dots, F_{i-1}$ incident to F_i
  2) an initial shelling of F_i (by IH(d-1) part (ii))

$\square$

NOTE: replacing c by $-c$ shows that $F_m, \dots, F_1$ is a shelling as well

$\longrightarrow$ in fact, this is true for any shelling of a polytope

Ex: if $F_1, \dots, F_m$ is a shelling of a polytope, so is $F_m, \dots, F_1$ (show this).

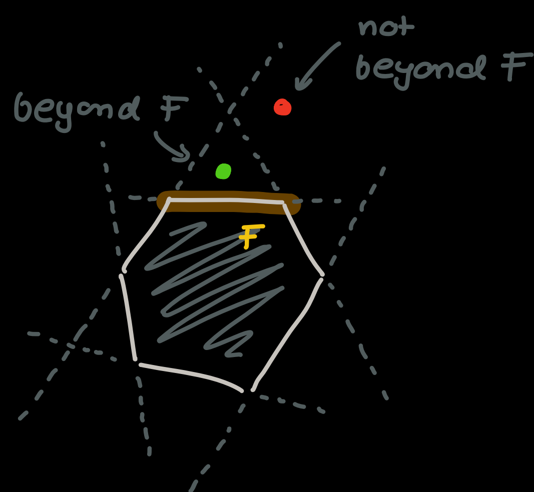
- one can use this to prove the Dehn-Sommerville equations.

Let's discuss some other uses of complexes

5.5. Schlegel diagrams

= Visualization technique for 4-polytopes

Def: a point $x \in \mathbb{R}^d$ lies **beyond** a facet $F \in \mathcal{F}_{d-1}(P)$ if it is "below" every facet-defining hyperplane except the one of F.



Def:

- fix a facet $F \in \mathcal{F}_{d-1}(P)$ and a point x beyond F.
- project every other face $F' \in \mathcal{F}(P) \setminus \{P, F\}$ onto F via point projection towards x.

$\longrightarrow$ this yields a polytopal complex $\mathcal{C}$ with support F (=: polytopal subdivision of F)

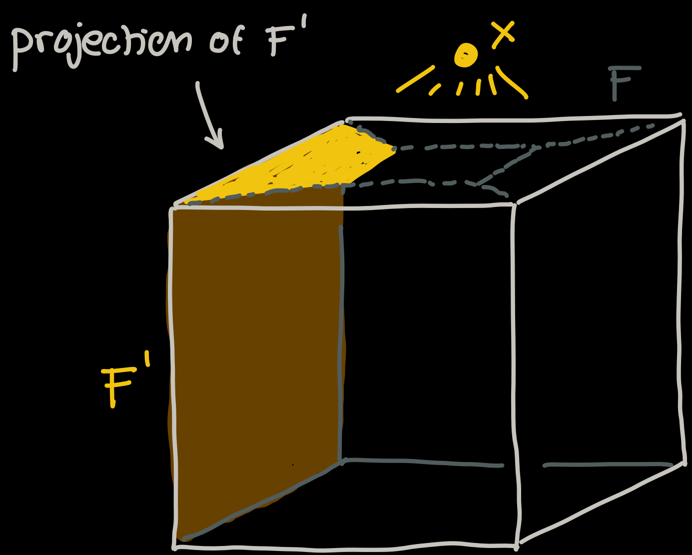
:= union of all faces

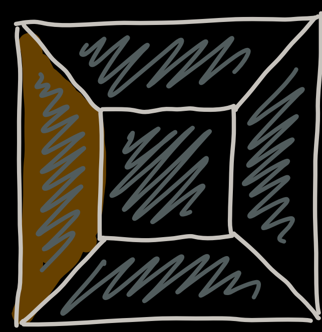
$\longrightarrow$ this is called a **Schlegel diagram** of P

- The full combinatorics of P can be reconstructed from each Schlegel diagram

Ex: Schlegel diagrams are shellable.

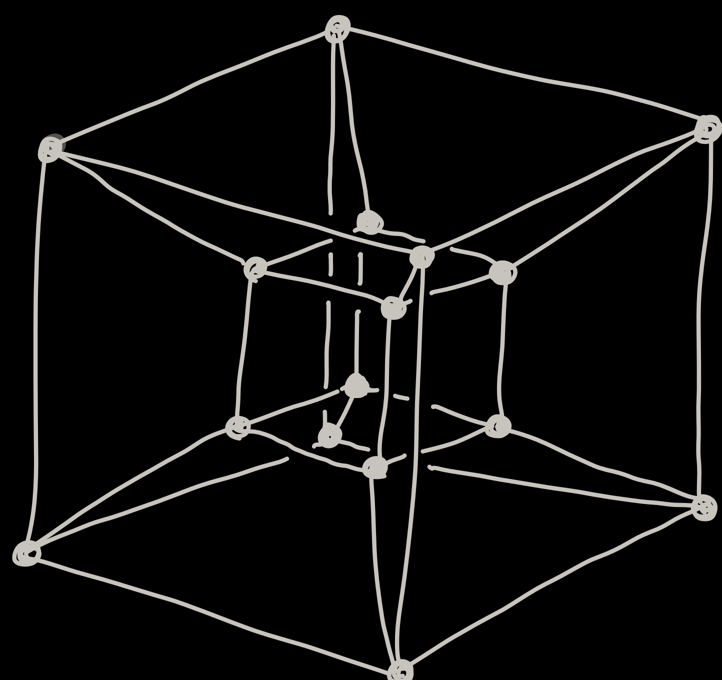
Examples: • cube





Schlegel diagram of 3-cube
= subdivision of square

• 4-cube

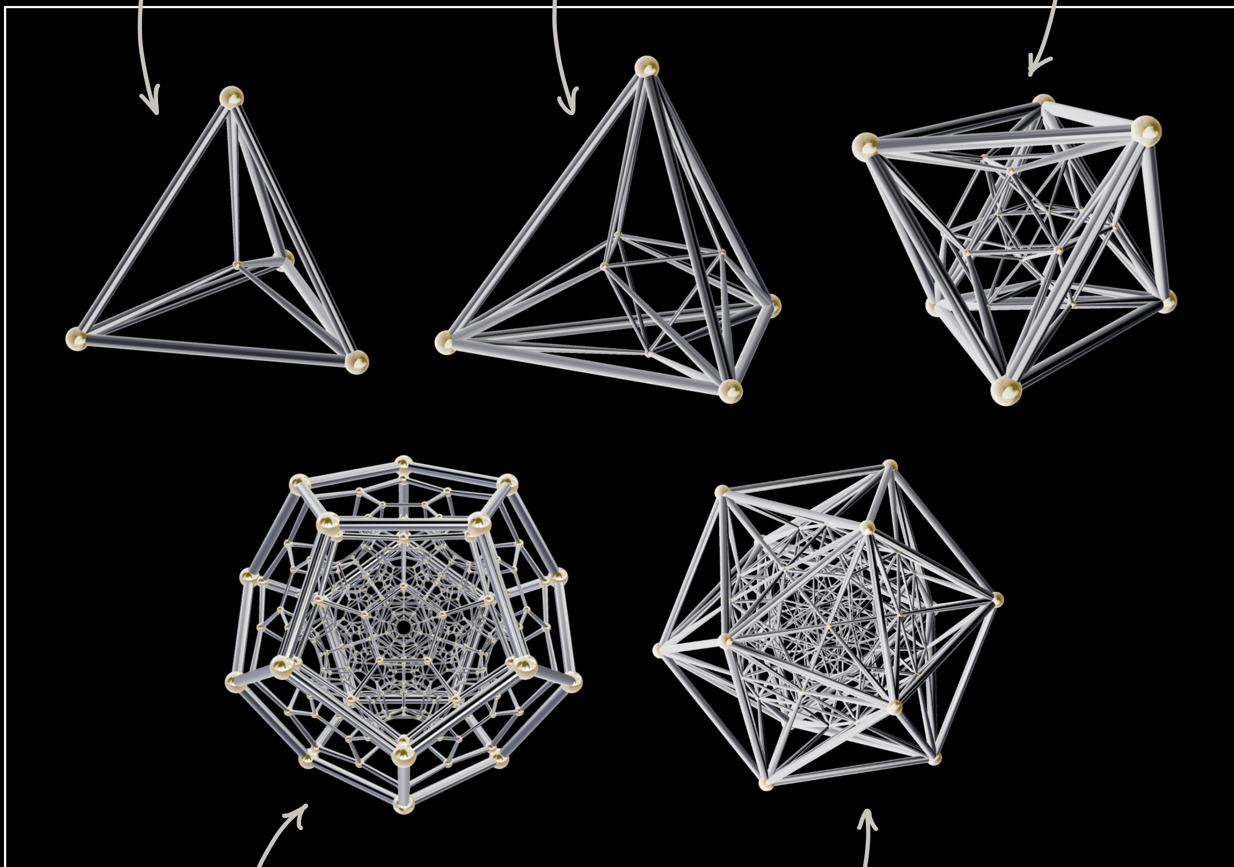


= subdivision of a 3-cube into 7 combinatoriel cubes

(polytopes comb. equiv. to cubes)

- the other reguler 4-polytopes:

4-simplex 4-crosspolytope 24-cell



[Wikipedia]

120-cell 600-cell

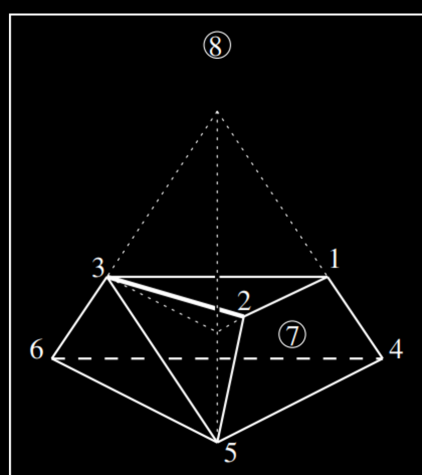
Ex: draw Schlegel diagram of tetrahedron prism

= cartesian product of tetrahedron and line segment

Q: is every subdivision of a 3-polytope a Schlegel-diagram?

→ **NO**: see Ziegler

building block of non-Schlegel subdivision

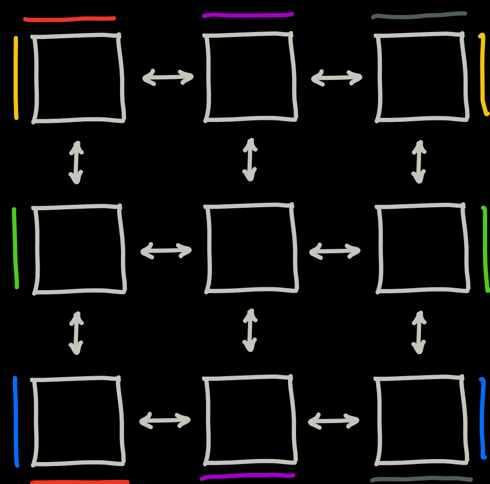


[Ziegler]

5.6. Abstract complexes & polytopal spheres

An abstract polytopal complex can consist of polytopes that do not necessarily live in the same ambient space. We identify their facets abstractly.

Example



- identify edges along arrows and same-colored edges

⟶ torus

- is not necessarily embedded into any Euclidean space.

- A polytopal sphere is an abstract complex homeomorphic to a sphere

E.g. ∂P is a polytopal sphere

BUT not every polytopal sphere comes from a polytope!

Ex: construct such a sphere from a non-Schlegel subdivision.

- A simplicial complex resp. sphere is a polytopal complex resp. sphere where every face is a simplex.

  E.g. P simplicial polytope
  $\longrightarrow$ ∂P is a simplicial sphere

Some facts about simplicial spheres: (not included in the lecture)

- not every simplicial sphere comes from a polytope
  - smallest example: $d=4$, $f_0 = 8$
  - OPEN: Does every simplicial sphere come from a non-convex polytope? (probably not)

- it is algorithmically undecidable whether a simplicial complex is a simplicial sphere (in dimension ≥ 5)
- many combinatorial results for simplicial polytopes extend to spheres (very non-trivially)
  - Dehn-Sommerville equations
  - upper bound theorem
  - g-theorem

$\longrightarrow$ philosophical point: these results are more about being spherical than about being convex.